We show that every state $\omega$ on a lattice effect algebra $E$ induces a uniform topology on $E$. If $\omega$ is subadditive this topology coincides with pseudometric topology induced by $\omega$. Further, we show relations between the interval and order topology on $E$ and topologies induced by states.

Keywords: lattice effect algebra, state, valuation, interval topology, order topology, topology induced by a state

1. Introduction and basic definitions

A model for an effect algebra is the standard effect algebra of positive self-adjoint operators dominated by the identity on a Hilbert space. In general an effect algebra is a partial algebra satisfying very simple axioms.

Effect algebras [6] (or, equivalent in some sense, $D$-posets [13], [14]) were introduced as carriers of states or probability measures in the quantum (or fuzzy) probability theory (see [10], [11], [13]). Thus elements of these structures represent quantum effects or fuzzy events which have yes-no character that may be unsharp or imprecise. Unfortunately, there are even finite effect algebras admitting no states hence also no probabilities [19]. Moreover, a state on an effect algebra need not be subadditive. It was proved in [20] that a state on a lattice effect algebra is subadditive iff it is a valuation. Further, if a faithful (i.e., non-zero at non-zero elements) valuation on an effect algebra $E$ exists then $E$ is modular and separable [20]. Conversely, on every complete modular atomic effect algebra there exists an ($\alpha$)-continuous state [18], [21]. The aim of this paper is to bring some topological properties of lattice (or complete) effect algebras on which states, order-continuous states or valuations exist. Namely, we study properties of order and interval topologies of such effect algebras. Further we show relations of these topologies to uniform or metric topologies induced by states or valuations on them.
Definition 1.1. A structure \((E; \oplus, 0, 1)\) is called an effect-algebra if 0, 1 are two distinguished elements and \(\oplus\) is a partially defined binary operation on \(P\) which satisfies the following conditions for any \(a, b, c \in E\):

(i) \(b \oplus a = a \oplus b\) if \(a \oplus b\) is defined,
(ii) \((a \oplus b) \oplus c = a \oplus (b \oplus c)\) if one side is defined,
(iii) for every \(a \in P\) there exists a unique \(b \in P\) such that \(a \oplus b = 1\) (we put \(a' = b\)),
(iv) if \(1 \oplus a\) is defined then \(a = 0\).

We often denote the effect algebra \((E; \oplus, 0, 1)\) briefly by \(E\). In every effect algebra \(E\) we can define the partial operation \(\ominus\) and the partial order \(\leq\) by putting

\[
a \leq b \text{ and } b \ominus a = c \text{ if } a \oplus c \text{ is defined and } a \oplus c = b.
\]

Since \(a \ominus c = a \ominus d\) implies \(c = d\), the \(\ominus\) and the \(\leq\) are well defined. If \(E\) with the defined partial order is a lattice (a complete lattice) then \((E; \ominus, 0, 1)\) is called a lattice effect algebra (a complete effect algebra). It is well known that a lattice effect algebra is a common generalization of orthomodular lattices and MV-algebras (see [4] and [14]).

Lemma 1.2. Elements of an effect algebra \((E; \oplus, 0, 1)\) satisfy the properties:

(i) \(a \oplus b\) is defined iff \(a \leq b'\),
(ii) \(a \leq a \oplus b\),
(iii) if \(a \oplus b\) and \(a \lor b\) exist then \(a \land b\) exists and \(a \oplus b = (a \land b) \ominus (a \lor b)\),
(iv) \(a \oplus b \leq a \ominus c\) iff \(b \leq c\) and \(a \ominus c\) is defined,
(v) \(a \ominus b = 0\) iff \(a = b\),
(vi) \(a \leq b \leq c\) implies that \(c \ominus b \leq c \ominus a\) and \(b \ominus a = (c \ominus a) \ominus (c \ominus b)\).

If \(E\) is a lattice effect algebra then

(vii) \(c \leq a, b \implies (a \lor b) \ominus c = (a \ominus c) \lor (b \ominus c)\) and \((a \land b) \ominus c = (a \ominus c) \land (b \ominus c)\),
(viii) \(a, b \leq c \implies c \ominus (a \lor b) = (c \ominus a) \lor (c \ominus b)\) and \(c \ominus (a \land b) = (c \ominus a) \land (c \ominus b)\),
(ix) \(a, b \leq c' \implies (a \ominus c) \lor (b \ominus c) = (a \lor b) \ominus c\) and \((a \land b) \ominus c = (a \ominus c) \land (b \ominus c)\).

It is worth noting that if \((E; \oplus, 0, 1)\) is an effect algebra then \((E; \ominus, 0, 1)\) with the partial binary operation \(\ominus\) defined above is a D-poset introduced by Kópka and Chovanec [14], and vice versa.

Definition 1.3. Let \((E; \ominus, 0, 1)\) be an effect algebra. \(Q \subseteq E\) is called a sub-effect algebra if

(i) \(1 \in Q\),
(ii) if from elements \(a, b, c \in E\) with \(a \ominus b = c\) at least two are elements of \(Q\) then \(a, b, c \in Q\).

For more details on D-posets and effect algebras we refer the reader to [4].

Definition 1.4. Assume that \((E; \oplus, 0, 1)\) is an effect algebra. A map \(m : E \to [0, 1]\) is called a (finitely additive) state on \(E\) if \(m(1) = 1\) and \(a \leq b' \implies m(a \oplus b) = m(b)\).
\(m(a) + m(b)\). We say that \(m\) is faithful if \(m(a) = 0 \implies a = 0\).

A state \(m\) on a lattice effect algebra \(E\) is called a valuation if for \(a, b \in E\), \(a \land b = 0 \implies m(a \lor b) = m(a) + m(b)\).

Note that if \(m\) is a state on an effect algebra \(E\) then for \(a, b \in E\) with \(a \leq b\) we have \(b = a \oplus (b \odot a)\), which implies \(m(b) = m(a) + m(b \odot a)\). Thus \(a \leq b \implies m(a) \leq m(b)\) and \(m(b \odot a) = m(b) - m(a)\).

If \(\omega\) is a valuation on a lattice effect algebra \(E\) then evidently \(\omega(a \lor b) \leq \omega(a) + \omega(b)\) for all \(a, b \in E\) (we say that \(\omega\) is subadditive). On the other hand a state on a lattice effect algebra need not be subadditive.

**Theorem 1.5.** [20] Assume that \(E\) is a lattice effect algebra.

(i) Every subadditive state \(\omega\) on \(E\) is a valuation.

(ii) A state \(\omega\) on \(E\) is a valuation iff \(\omega(a \lor b) + \omega(a \land b) = \omega(a) + \omega(b)\) for all \(a, b \in E\).

(iii) If there exists a faithful valuation \(\omega\) on \(E\) then \(E\) is modular and separable.

The existence of \((a)\)-continuous states or valuations on some families of lattice effect algebras has been proved in [18], [21], [22].

### 2. Uniform topologies induced by states on lattice effect algebras

If a net \((x_\alpha)_{\alpha \in \mathcal{E}}\) of elements of a topological space \((X, \tau)\) converges to a point \(x \in X\) we will write \(x_\alpha \xrightarrow{\tau} x\). Here \(\tau\) denotes also the collection of all open subsets of \(X\).

**Theorem 2.1.** Every state \(\omega\) on a lattice effect algebra \(E\) induces a uniform topology \(\tau_\omega\) such that for a net \((x_\alpha)_{\alpha \in \mathcal{E}}\) of elements of \(E\)

\[x_\alpha \xrightarrow{\tau_\omega} x \text{ iff } \omega(x_\alpha \lor y) \rightarrow \omega(x \lor y) \text{ and } \omega(x_\alpha \land y) \rightarrow \omega(x \land y) \text{ for all } y \in E.\]

**Proof.** Consider the function family \(\Phi = \{\omega_y \lor \mid y \in E\} \cup \{\omega_y \land \mid y \in E\}\), where \(\omega_y \lor : E \to [0, 1]\) and \(\omega_y \land : E \to [0, 1]\) are defined by putting \(\omega_y \lor (x) = \omega(y \lor x)\) and \(\omega_y \land (x) = \omega(y \land x)\) for all \(x \in E\). Further, consider the family of pseudometrics on \(E\): \(\Sigma_\Phi = \{\rho_y \lor \mid y \in E\} \cup \{\rho_y \land \mid y \in E\}\), where \(\rho_y \lor (a, b) = |\omega_y \lor (a) - \omega_y \lor (b)|\) and \(\rho_y \land (a, b) = |\omega_y \land (a) - \omega_y \land (b)|\) for all \(a, b \in E\). Let us denote by \(U_\Phi\) the uniformity on \(E\) induced by the family of pseudometrics \(\Sigma_\Phi\). Further denote by \(\tau_\omega\) the topology compatible with the uniformity \(U_\Phi\). Then for every net \((x_\alpha)_{\alpha \in \mathcal{E}}\) of elements of \(E\)

\[x_\alpha \xrightarrow{\tau_\omega} x \text{ iff } \omega(x_\alpha \lor y) \rightarrow \omega(x \lor y) \text{ and } \omega(x_\alpha \land y) \rightarrow \omega(x \land y) \text{ for all } y \in E.\]
For a deeper discussion of a topology induced by a function family \( \Phi \) we refer the reader to [3]. In [15] functions on \( D \)-posets with values in arbitrary uniform space without algebraic operations were treated and a Nikodym boundedness theorem and a convergence theorem were proved.

3. Pseudometric topologies on lattice effect algebras induced by subadditive states

For elements \( a, b \) of a lattice effect algebra \( E \) we set \( a \triangle b = (a \lor b) \cap (a \land b) \). Then the triangle inequality \( a \land b \leq (a \triangle b) \lor (a \land b) \) fails to be true in general but it does so for every valuation \( \omega \) on \( E \).

**Lemma 3.1.** For every valuation \( \omega \) on a lattice effect algebra \( E \) and for all \( a, b, c \in E \), \( \omega(a \land b) \leq \omega(a \lor c) + \omega(c \land b) \).

**Proof.** For any \( a, b, c \in E \) we have \( \omega(a \land c) + \omega(b \land c) = \omega((a \land c) \lor (b \land c)) + \omega(a \land b \land c) \).

Moreover, \( (a \land c) \lor (b \land c) \leq (c \lor (a \lor b)) \) which gives \( \omega((a \land c) \lor (b \land c)) - \omega((c \lor (a \lor b)) \leq 0 \). Therefore \( \omega(a \land b) = \omega(a \lor b) - \omega(a \land b) \leq \omega(a \lor b \lor c) - \omega(a \land b \land c) = \omega(a \lor c) + \omega(b \lor c) - \omega((a \lor c) \lor (b \lor c)) - \omega(a \land c) - \omega(b \land c) + \omega((a \lor c) \lor (b \lor c)) = \omega(a \lor c) + \omega(b \lor c) + \omega((a \lor c) \lor (b \lor c)) - \omega((a \lor c) \lor (b \lor c)) \leq \omega(a \lor c) + \omega(b \lor c) \).

Assume that \( (\mathcal{E}; \prec) \) is a directed set and \( (P; \leq) \) is a poset. A net of elements of \( P \) is denoted by \( (a_\alpha)_{\alpha \in \mathcal{E}} \). If \( a_\alpha \preceq a_\beta \) for all \( \alpha, \beta \in \mathcal{E} \) such that \( \alpha \prec \beta \) then we write \( a_\alpha \uparrow \). If moreover \( a = \bigvee \{a_\alpha \mid \alpha \in \mathcal{E} \} \) we write \( a_\alpha \uparrow a \). The meaning of \( a_\alpha \downarrow \) and \( a_\alpha \downarrow a \) is dual. For instance, \( a \uparrow u_\alpha \leq v_\alpha \downarrow b \) means that \( u_\alpha \preceq v_\alpha \) for all \( \alpha \in \mathcal{E} \) and \( u_\alpha \uparrow a \) and \( v_\alpha \downarrow b \). We will write \( b \preceq a_\alpha \uparrow a \) if \( b \preceq a_\alpha \) for all \( \alpha \in \mathcal{E} \) and \( a_\alpha \uparrow a \).

A net \( (a_\alpha)_{\alpha \in \mathcal{E}} \) of elements of a poset \( (P; \leq) \) order converges to a point \( a \in P \) if there are nets \( (u_\alpha)_{\alpha \in \mathcal{E}} \) and \( (v_\alpha)_{\alpha \in \mathcal{E}} \) of elements of \( P \) such that

\[
a_\alpha \uparrow a_\alpha \leq a_\alpha \leq v_\alpha \downarrow a.
\]

We write \( a_\alpha \xrightarrow{(o)} a \) in \( P \) (or briefly \( a_\alpha \xrightarrow{(o)} a \)).

**Lemma 3.2.** In every lattice effect algebra \( E \)

\[
x_\alpha \xrightarrow{(o)} x \text{ if } x_\alpha \land x \xrightarrow{(o)} 0; \quad x_\alpha, x \in E.
\]

**Proof.** (1) Evidently \( x_\alpha \xrightarrow{(o)} x \Rightarrow x_\alpha \lor x \xrightarrow{(o)} x \) and \( x_\alpha \land x \xrightarrow{(o)} x \). By the definition of \( (o)\)-convergence there are nets \( (u_\alpha)_{\alpha \in \mathcal{E}}, (v_\alpha)_{\alpha \in \mathcal{E}} \) such that \( x \uparrow u_\alpha \leq x_\alpha \land x \leq x_\alpha \lor x \leq v_\alpha \downarrow x \) which implies that \( x_\alpha \land x = (x_\alpha \lor x) \cap (x_\alpha \land x) \leq v_\alpha \lor u_\alpha \downarrow 0 \) (see [16]) and hence \( x_\alpha \land x \xrightarrow{(o)} 0 \).
Theorem 3.3. For a state $\omega$ on a lattice effect algebra $E$ the following conditions are equivalent:

(i) $\omega$ is subadditive,
(ii) $\omega$ is a valuation,
(iii) $\rho_\omega: E \times E \to [0, 1]$ defined by $\rho_\omega(a, b) = \omega(a \triangle b)$ is a pseudometric.

Proof For a proof of (i) $\iff$ (ii) we refer to [20].

(i) $\implies$ (iii): By Lemma 3.1, $\rho_\omega(a, b) \leq \rho_\omega(a, c) + \rho_\omega(b, c)$ for all $a, b, c \in E$. The rest is trivial.

(iii) $\implies$ (ii): If $\rho_\omega$ is a pseudometric then $\omega(a \triangle b) = \omega(a \lor b) - \omega(a \land b) = \rho_\omega(a, b) \leq \rho_\omega(a, a \land b) + \rho_\omega(a \land b, b) = \omega(a) - \omega(a \land b) + \omega(b) - \omega(a \land b)$ which gives $\omega(a \lor b) \leq \omega(a) + \omega(b)$.

In the sequel, we will denote by $\tau_{\rho_\omega}$ the pseudometric topology compatible with $\rho_\omega$.

It is easy to check that $\rho_\omega$ has the following properties:

(1) $\rho_\omega(0, a) = \rho_\omega(b, a \lor b)$, for all $b \leq a'$
(2) $0 \leq a \leq b \implies \rho_\omega(0, b) = \rho_\omega(0, a) + \rho_\omega(a, b)$
(3) $\rho_\omega(a, b) = \rho_\omega(a \land b, a \lor b) = \rho_\omega(a', b')$

which gives

(4) $a_\alpha \xrightarrow{\tau_{\rho_\omega}} a$ iff $a'_\alpha \xrightarrow{\tau_{\rho_\omega}} a'$
(5) $a_\alpha \xrightarrow{\tau_{\rho_\omega}} 0$ iff $a_\alpha \perp b \xrightarrow{\tau_{\rho_\omega}} b$ for all $a_\alpha \leq b'$.

By (5) we obtain

(6) If $b \leq b_\alpha \leq c$ then $b_\alpha \xrightarrow{\tau_{\rho_\omega}} b$ iff $b_\alpha \perp b \xrightarrow{\tau_{\rho_\omega}} b$ for all $b_\alpha \perp c$.

Theorem 3.4. For every valuation $\omega$ on a lattice effect algebra $E$, $\tau_{\rho_\omega} = \tau_\omega$.

Proof (1) Assume $a_\alpha \xrightarrow{\tau_{\rho_\omega}} a$. Then for every $x \in E$ we have $\omega(a_\alpha \lor x) \to \omega(a \lor x)$ and $\omega(a_\alpha \land x) \to \omega(a \land x)$, hence $\omega(a_\alpha \triangle a) = \omega(a_\alpha \lor a) - \omega(a_\alpha \land a) \to \omega(a) - \omega(a)$ is equivalent to $a_\alpha \xrightarrow{\tau_\omega} a$.

(2) Conversely, let $a_\alpha \xrightarrow{\tau_\omega} a$. Then $\omega(a_\alpha \triangle a) = (\omega(a_\alpha \lor a) - \omega(a)) + (\omega(a) - \omega(a_\alpha \land a)) \to 0$ which gives $\omega(a_\alpha \lor a) \to \omega(a)$ and $\omega(a_\alpha \land a) \to \omega(a)$. It follows that also $\omega(a_\alpha) \to \omega(a)$. Let $x \in E$ be arbitrary. Then $\omega(a_\alpha \lor x) \leq \omega(a_\alpha \lor a) + \omega(x) - \omega(a \land x) \to \omega(a) + \omega(x) - \omega(a \land x)$, which gives $\omega(a_\alpha \lor a \lor x) \to \omega(a \lor x)$. Further, $\omega(a_\alpha \lor a) + \omega(x) - \omega(a \land x) \leq \omega((a_\alpha \lor a) \land x) \leq \omega(a_\alpha \lor x) \leq \omega(a_\alpha \lor a \lor x) \to \omega(a \lor x)$ and hence $\omega(a_\alpha \lor x) \to \omega(a \lor x)$. Moreover,
consider that continuous for brevity) if for any net of elements of $E$

It is easily seen that in an $(\omega, \tau)$-continuous lattice effect algebra we have:

$$\alpha \in \tau \text{ iff for every net } (x_\alpha)_{\alpha \in \mathcal{E}} \text{ of elements of } F: x_\alpha \to x \implies x_\alpha \in \tau.$$  

We need only consider that $x_\alpha \uparrow x$ iff $x'_\alpha \uparrow x'$ and that in every lattice $x_\alpha \cup y \uparrow x \cup y$. Note that $(\omega, \tau)$-continuous lattice effect algebras were also called meet continuous lattices [8].

**Theorem 4.2.** Let $E$ be a lattice effect algebra and let $\omega: E \to [0, 1]$ be a state on $E$. Then

(i) $\omega$ is faithful $\implies \tau_\omega$ is $T_2$,

(ii) $\omega$ is faithful $\implies \tau_1 \subseteq \tau_\omega$,

(iii) $\omega$ is subadditive and faithful $\implies \tau_\omega = \tau_{\rho_\omega}$ is a metric topology,

(iv) $\omega$ is subadditive and faithful $\implies (\tau_{\rho_\omega} \subseteq \tau_\omega$ iff $\omega$ is $(\omega, \tau)$-continuous),

(v) $E$ and $\omega$ are $(\omega, \tau)$-continuous $\implies \tau_\omega \subseteq \tau_0$.

**Proof** (i) Assume that $x_\alpha \to x_1$ and $x_\alpha \to x_2$. If $x_1 \neq x_2$ then either $x_1 \wedge x_2 \neq x_1$ or $x_1 \vee x_2 \neq x_2$. Let $x_1 \wedge x_2 \neq x_1$. Then $\omega(x_1 \wedge x_2) < \omega(x_1)$. By definition of $\tau_\omega$ we have $\omega(x_\alpha \wedge x_1) \to \omega(x_1)$ and $\omega(x_\alpha \wedge x_2) \to \omega(x_2 \wedge x_1)$, a contradiction. Hence $x_1 = x_2$, which gives that $\tau_\omega$ is $T_2$.

(ii) Let $a \leq x \leq b$ and let $x_\alpha \to x$. Then $x_\alpha = x, a = x_\alpha \vee a$ and by definition of $\tau_\omega$ we have $\omega(x_\alpha) = \omega(x_\alpha \vee a) = \omega(x_\alpha \wedge b) \to \omega(x) = \omega(x \vee a) = \omega(x \wedge b)$. As $x \wedge b \leq x \wedge a$ and $\omega$ is faithful we conclude that $x \wedge b = x \vee a$ which gives $x \in [a, b]$.

(iii) As $\omega$ is a faithful valuation, we have $\tau_\omega = \tau_{\rho_\omega}$ and $\omega(x \triangle y) = 0$ iff $x \triangle y = 0$ iff $(x \vee y) \cap (x \wedge y) = 0$ iff $x \wedge y = x \vee y$ iff $x = y$, which gives $\rho_\omega(x, y) = 0$ iff $x = y$.

(iv) Assume that $\omega$ is $(\omega, \tau)$-continuous. Then by Lemma 3.2, $x_\alpha \to x \implies x_\alpha \triangle x \to 0 \implies \omega(x_\alpha \triangle x) \to \omega(0) = 0 \implies x_\alpha \to x$. It follows by definition of $\tau_\omega$ that $\tau_{\rho_\omega} \subseteq \tau_\omega$. 

\[ \omega(a \wedge x) = \omega(a) + \omega(x) - \omega(a \vee x) \to \omega(a) + \omega(x) - \omega(a \vee x) = \omega(a \wedge x). \]
Conversely, if \( \tau_\rho \subseteq \tau_\omega \) then \( x_\alpha \xrightarrow{(o)} x \implies x_\alpha \xrightarrow{\tau_\omega} x \implies x_\alpha \xrightarrow{(o)} x \). As \( \omega \) is \( \tau_\omega \)-continuous and \( \tau_\omega = \tau_\rho \), we conclude that \( \omega(x_\alpha) \rightarrow \omega(x) \), hence \( \omega \) is \( (o) \)-continuous.

(v) Assume that \( x_\alpha \xrightarrow{(o)} x \). Then by \( (o) \)-continuity of \( E \) we have \( x_\alpha \lor y \xrightarrow{(o)} x \lor y \) and \( x_\alpha \land y \xrightarrow{(o)} x \land y \) for all \( y \in E \). By \( (o) \)-continuity of \( \omega \) we obtain \( \omega(x_\alpha \lor y) \rightarrow \omega(x \lor y) \) and \( \omega(x_\alpha \land y) \rightarrow \omega(x \land y) \), for all \( y \in E \), which gives \( x_\alpha \xrightarrow{\tau_\omega} x \). It follows \( \tau_\omega \subseteq \tau_\rho \) by definition of \( \tau_\rho \).

**Definition 4.3.** We say that a bounded lattice \( L \) has separated intervals if given any two disjoint intervals \([a, b], [c, d] \subseteq L\), the lattice \( L \) can be covered by finite number of closed intervals each of which is disjoint with at least one of \([a, b]\) and \([c, d]\).

In [17] it was proved that the interval topology \( \tau_i \) on a complete lattice \( L \) is Hausdorff if \( L \) has separated intervals.

Since the partial operation \( \oplus \) on an effect algebra \( E \) is associative, the existence and the meaning of \( a_1 \oplus a_2 \oplus \cdots \oplus a_n \) for elements of \( E \) is defined recurrently. \( M \subseteq E \) is called an orthogonal set if for every finite set \( \{a_1, a_2, \ldots, a_n\} \subseteq M \), \( a_1 \oplus a_2 \oplus \cdots \oplus a_n \) is defined. \( Q \subseteq E \) is called a set of mutually orthogonal elements if any two different elements \( a, b \in Q \) are orthogonal; i.e., \( a \leq b' \). Evidently, every orthogonal set is a set of mutually orthogonal elements but not conversely.

**Definition 4.4.** An effect algebra \((E; \oplus, 0, 1)\) is called Archimedean if for no nonzero element \( e \in E \), \( ne = e \oplus e \oplus \cdots \oplus e \) \((n\text{-times})\) exists for every \( n \in \mathbb{N} \). \( E \) is called separable if it is Archimedean and every orthogonal set of elements in \( E \) is at most countable.

It was proved in [20] that if there exists a faithful state \( m \) on an effect algebra \((E; \oplus, 0, 1)\) then \( E \) is separable.

**Theorem 4.5.** Let \( E \) be a complete effect algebra with separated intervals.

(i) If there exists a faithful, subadditive and \( (o) \)-continuous state \( \omega \) on \( E \) then

\[ \tau_i = \tau_\omega = \tau_\rho = \tau_\omega \]

and \((E, \tau_\rho)\) is a compact metric space. Moreover, \( E \) is \( (o) \)-continuous, modular and separable.

(ii) If there exists an \( (o) \)-continuous state on \( E \) and \( E \) is \( (o) \)-continuous then

\[ \tau_i = \tau_\omega = \tau_\omega \]

and \( \tau_\omega \) is a compact Hausdorff topology on \( E \).
Proof (i) By Theorem 4.2 we have \( \tau_i \subseteq \tau_\omega \subseteq \tau_\rho \omega \subseteq \tau_o \). As \( E \) is a complete lattice with separated intervals, the topology \( \tau_i \) is \( T_2 \) [17]. It follows that \( \tau_i = \tau_o \) and by the Frink theorem \( \tau_i \) is compact as \( E \) is complete. For the proof that \( E \) is \((o)\)-continuous, modular and separable, we refer the reader to [20].

(ii) By Theorem 4.2, \( \tau_i \subseteq \tau_\omega \subseteq \tau_o \) and by [17] \( \tau_i = \tau_o \).

Remark 4.6. Note that the \((o)\)-continuity of a state \( \omega \) on a separable lattice effect algebra \( E \) is equivalent with \( \sigma \)-additivity of \( \omega \), i.e., \( \omega(\bigoplus_{n=1}^{\infty} x_n) = \sum_{n=1}^{\infty} \omega(x_n) \), where \( \bigoplus_{n=1}^{\infty} x_n = \bigvee_{n=1}^{\infty} (\bigoplus_{k=1}^{n} x_k) \) for every sequence \( (x_n)_{n=1}^{\infty} \) in \( E \) for which \( \bigvee_{n=1}^{\infty} (\bigoplus_{k=1}^{n} x_k) \) is defined.

In [22] it was proved that on every Archimedean atomic distributive effect algebra \( E \) (e.g., every Archimedean atomic \( MV \)-algebra) there exists an \((o)\)-continuous subadditive state. By [18] on every separable complete modular atomic effect algebra \( E \) there exists an \((o)\)-continuous faithful state. Moreover, by a generalization of the Kaplansky theorem [18] every complete modular atomic effect algebra is an \((o)\)-continuous lattice.

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