ORDER-TOPOLOGICAL LATTICE EFFECT ALGEBRAS

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Abstract. We study order convergence of nets in lattice effect algebras, which generalized orthomodular lattices, including Boolean algebras and MV-algebras in quantum or fuzzy probability theory. We show that in a complete atomic \((o\)-continuous effect algebra \(E\) the order convergence of nets is topological if and only if the order topology on \(E\) is Hausdorff. If moreover \(E\) is distributive (e.g., MV-algebra) then the order topology is compact Hausdorff.

1. Introduction and basic definitions

Effect algebras, or equivalent in some sense D-posets were introduced as carriers of probability measure in quantum or fuzzy probability theory. Elements of these structures represent quantum effects or fuzzy events that may be unsharp or imprecise ([6], [13]). Lattice ordered effect algebras generalize orthomodular lattices [12] including Boolean algebras and MV-algebras [1], [2], [10], [11], [14].

Definition 1. [6]. A structure \((E; \oplus, 0, 1)\) is called an effect-algebra if 0, 1 are two distinguished elements and \(\oplus\) is a partially defined binary operation on \(P\) which satisfies the following conditions for any \(a, b, c \in E\):

(i) \(b \oplus a = a \oplus b\) if \(a \oplus b\) is defined,
(ii) \((a \oplus b) \oplus c = a \oplus (b \oplus c)\) if one side is defined,
(iii) for every \(a \in P\) there exists a unique \(b \in P\) such that \(a \oplus b = 1\) (we put \(a' = b\)),
(iv) if \(1 \oplus a\) is defined then \(a = 0\).

We often denote the effect algebra \((E; \oplus, 0, 1)\) briefly by \(E\). In every effect algebra \(E\) we can define the partial operation \(\ominus\) and the partial order \(\leq\) by putting

\[a \leq b\text{ and }b \ominus a = c\text{ iff }a \ominus c\text{ is defined and }a \ominus c = b.\]

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Since $a \oplus c = a \oplus d$ implies $c = d$, the $\oplus$ and the $\leq$ are well defined. If $E$ with the defined partial order is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a lattice effect algebra (a complete effect algebra). If $(E; \oplus, 0, 1)$ is an effect algebra then $(E; \ominus, 0, 1)$ with the partial binary operation $\ominus$ defined above is a $D$-poset, introduced by Kôpka [13] as a new algebraic structure of fuzzy sets, and vice versa, [14]. For more details on $D$-posets and effect algebras we refer the reader to [3], [15].

**Definition 2.** Elements $a$ and $b$ of a lattice effect algebra $E$ are called compatible (written $a \leftrightarrow b$) if $a \vee b = a \oplus (b \ominus (a \wedge b))$, [14].

On many places we will need the following statement proved in [11].

**Lemma 3.** Let $E$ be a lattice effect algebra and $A \subseteq E$ with $\bigvee A$ existing in $E$. If $b \in E$ is compatible with every $a \in A$ then $b \leftrightarrow \bigvee A$ and $b \wedge (\bigvee A) = \bigvee \{a \wedge b \mid a \in E\}$.

A lattice effect algebra is called modular or distributive if $E$ as a lattice has these properties [9]. A lattice effect algebra is called an MV-effect algebra if every two elements $a, b \in E$ are compatible. It has been shown by Kôpka and Chovanec [14] that an MV-effect algebra $E$ can be organized into an MV-algebra and vice versa.

## 2. Topological Effect Algebras

Assume that $(\mathcal{E}; \prec)$ is a directed set and $(P; \leq)$ is a poset. A net of elements of $P$ is denoted by $(a_\alpha)_{\alpha \in \mathcal{E}}$. If $a_\alpha \leq a_\beta$ for all $\alpha, \beta \in \mathcal{E}$ such that $\alpha \prec \beta$ then we write $a_\alpha \downarrow$. If moreover $a = \bigvee\{a_\alpha \mid \alpha \in \mathcal{E}\}$ we write $a_\alpha \downarrow a$. The meaning of $a_\alpha \downarrow$ and $a_\alpha \downarrow a$ is dual. For instance, $a \uparrow u_\alpha \leq v_\alpha \downarrow b$ means that $u_\alpha \leq v_\alpha$ for all $\alpha \in \mathcal{E}$ and $u_\alpha \uparrow a$ and $v_\alpha \downarrow b$. We will write $b \leq a_\alpha \uparrow a$ if $b \leq a_\alpha$ for all $\alpha \in \mathcal{E}$ and $a_\alpha \uparrow a$.

A net $(a_\alpha)_{\alpha \in \mathcal{E}}$ of elements of a poset $(P; \leq)$ order converges ((o)-converges, for short) to a point $a \in P$ if there are nets $(u_\alpha)_{\alpha \in \mathcal{E}}$ and $(v_\alpha)_{\alpha \in \mathcal{E}}$ of elements of $P$ such that

$$a \uparrow u_\alpha \leq a_\alpha \leq v_\alpha \downarrow a.$$  

We write $a_\alpha \xrightarrow{(o)} a$ in $P$ (or briefly $a_\alpha \xrightarrow{(o)} a$).

The strongest (biggest) topology on a poset $(P; \leq)$ such that (o)-convergence of nets of elements of $P$ implies topological convergence is called order topology ((o)-topology) on $P$ and it is denoted by $\tau_o$. The order sequence topology denoted by $\tau_{os}$ is the strongest topology on $P$ such that (o)-convergence of sequences implies topological convergence. We can show that $F \subseteq P$ is $\tau_o$-closed ($\tau_{os}$-closed) set if $F$ includes (o)-limits of all order convergent nets (sequences) of elements of $F$. In spite of that, the (o)-convergence and $\tau_o$-convergence of nets in (even complete) lattices need not coincide. Moreover, the fact that in
a lattice $L$ the order convergence of filters is topological does not imply the same statement for order convergence of nets as we have shown in [24].

For complete orthomodular lattices (including Boolean algebras) it has been shown in [24] and [4] that they are $(o)$-topological, i.e., order continuous lattices in which the order convergence of nets is identical with their convergence in the order topology iff they are atomic and $(o)$-continuous lattices.

We will be concerned with the above mentioned problem for complete atomic effect algebras and net-theoretical convergences, because convergences of nets play important task in the probability (or measure) theory on these structures [23].

Recall that an arbitrary system $G = (a_\kappa)_{\kappa \in H}$ of not necessarily different elements of an effect algebra $E$ is called $\oplus$-orthogonal if for every finite set $K \subseteq H$ the element $\bigoplus \{a_\kappa \mid \kappa \in K\}$ exists in $E$. If $\bigvee \{ \bigoplus_{\kappa \in K} a_\kappa \mid K \subseteq H \text{ is finite}\}$ exists then we put $\bigoplus_{\kappa \in H} a_\kappa = \bigvee \{ \bigoplus_{\kappa \in K} a_\kappa \mid K \subseteq H \text{ is finite}\}$.

An Archimedean effect algebra $E$ is called separable if every $\oplus$-orthogonal systems of elements of $E$ is at most countable. More detailed these notions are discussed in [25].

Lemma 4. $\tau_o = \tau_{os}$ on every complete separable effect algebra $E$.

Proof. In view of definitions of $\tau_o$ and $\tau_{os}$ we have $\tau_o \subseteq \tau_{os}$, as for every sequence $(x_n)_{n=1}^\infty$ we have $x_n \overset{o}{\rightarrow} x$ implies $x_n \overset{o}{\rightarrow} x$. Let $F \subseteq E$ be a $\tau_{os}$-closed set and $(x_\alpha)_{\alpha \in \mathcal{E}}$ be a net of elements of $E$ such that $x_\alpha \overset{o}{\rightarrow} x \in E$. Since $E$ is complete and separable, by [20, Theorem 4.7] there are $\alpha_1 \leq \alpha_2 \leq \ldots$ in $\mathcal{E}$ such that $x_{\alpha_n} \overset{o}{\rightarrow} x$, hence $x \in F$ and $F$ is $\tau_o$-closed. It follows that $\tau_{os} \subseteq \tau_o$ and hence $\tau_o = \tau_{os}$. □

In the paper by an order topological lattice ($(o)$-topological, for short) we mean a lattice $L$ whose order convergence of nets of elements coincides with convergence in the order topology $\tau_o$ and makes lattice operations continuous. For a lattice $L$ a subset $D \subseteq L$ is called a full sub-lattice if for all $P, Q \subseteq D$ with $\bigvee P$ and $\bigwedge Q$ existing in $L$ we have $\bigvee P, \bigwedge Q \in D$.

For net-theoretical convergence we will need some statements concerning the relativizations. Note that, in general, for a complete lattice $L$ with order topology $\tau_o$ and its sublattice $D$ with order topology $\tau_o^D$, need not be $\tau_o^D = \tau_o \cap D$. Thus the fact that $L$ is $(o)$-topological (in the sense of net convergences) does not imply that $D$ is $(o)$-topological, even in the case when the convergence of filters is $(o)$-topological. All these facts have been shown in [24, Example 4.1].

Lemma 5. Let $D$ be a sublattice of a lattice $L$ and $\tau_o^D$ and $\tau_o$ be order topologies on $D$ and $L$, respectively. Let $D$ be $\tau_o$-closed. Then:

(i) $D$ is a full sublattice of $L$. 

(ii) If \( L \) is complete then \( D \) is complete as well and for \( x_\alpha, x \in D \):
\[
x_\alpha \overset{(o)}{\rightarrow} x \text{ (in } D \text{) iff } x_\alpha \overset{o}{\rightarrow} x \text{ (in } L \text{),}
\]
\[
x_\alpha \overset{\tau_D}{\rightarrow} x \text{ (in } D \text{) iff } x_\alpha \overset{\tau_D}{\rightarrow} x \text{ (in } L \text{),}
\]

(iii) If \( L \) is complete and order topological then \( D \) is order topological.

(iv) Let \( L \) be complete and order topological then \( D \) be a map such that for \( x_\alpha, x \in D \):
\[
x_\alpha \overset{(o)}{\rightarrow} x \text{ (in } D \text{) } \implies f(x_\alpha) \overset{(o)}{\rightarrow} f(x) \text{ (in } L \text{). Then for } y_\alpha, y \in D :
\]
\[
y_\alpha \overset{\tau_D}{\rightarrow} y \implies f(y_\alpha) \overset{\tau_D}{\rightarrow} f(y).
\]

Proof. (i) Assume that \( A \subseteq D \) and \( \bigvee A \) exists in \( E \). Set \( x_\alpha = \bigvee \alpha \), for all finite \( \alpha \subseteq A \). Then \( x_\alpha \in D \) and \( x_\alpha \uparrow \bigvee A \), which gives \( x_\alpha \overset{\tau_D}{\rightarrow} \bigvee A \), hence \( \bigvee A \in D \). Dually, if \( B \subseteq D \) and \( \bigwedge B \) exists in \( L \) then \( \bigwedge B \in D \).

(ii) As by (i) for each \( H \subseteq D \) we have \( \bigvee H, \bigwedge H \in D \) we obtain that for \( x_\alpha \in D, \alpha \in \mathcal{E} \) we have
\[
\bigvee_{\beta \in \mathcal{E} \alpha \geq \beta} x_\alpha = \bigwedge_{\beta \in \mathcal{E} \alpha \geq \beta} x_\alpha \text{ (in } L \text{) if } \bigvee_{\beta \in \mathcal{E} \alpha \geq \beta} x_\alpha = \bigwedge_{\beta \in \mathcal{E} \alpha \geq \beta} x_\alpha \text{ (in } D \text{)}
\]
which is equivalent to
\[
x_\alpha \overset{(o)}{\rightarrow} x \text{ (in } L \text{) iff } x_\alpha \overset{o}{\rightarrow} x \text{ (in } D \text{)}
\]
as \( D \) and \( L \) are complete lattices. It follows that \( F \subseteq D \) is \( \tau_D \)-closed iff \( F \) is \( \tau_o \)-closed. Thus for \( U \subseteq D \) we have \( U \in \tau_D \) iff \( L \setminus (D \setminus U) \in \tau_o \) and hence for \( x_\alpha, x \in D \) we have \( x_\alpha \overset{\tau_D}{\rightarrow} x \) iff \( x_\alpha \overset{\tau_o}{\rightarrow} x \) iff \( \tau_D = \tau_o \cap D \).

(iii) This is a consequence of (ii).

(iv) We have to prove that \( f \) is a continuous map of \( (D, \tau_D) \) into \( (L, \tau_o) \), since by (ii) for \( y_\alpha, y \in D \) we have \( y_\alpha \overset{\tau_D}{\rightarrow} y \) iff \( y_\alpha \overset{\tau_o}{\rightarrow} y \). Assume that \( F \subseteq L \) is \( \tau_o \)-closed and \( x_\alpha \in f^{-1}(F), \alpha \in \mathcal{E} \). Then \( f(x_\alpha) \in F \) and \( x_\alpha \overset{(o)}{\rightarrow} x \) (in \( D \)) implies \( f(x_\alpha) \overset{(o)}{\rightarrow} f(x) \) (in \( L \)) which gives \( f(x) \in F \). Hence \( x \in f^{-1}(F) \), which proves that \( f^{-1}(F) \) is \( \tau_D \)-closed.

Recall that a lattice effect algebra \( E \) is \((o)\)-continuous if for \( x_\alpha, x, y \in E \):
\[
x_\alpha \uparrow x \implies x_\alpha \wedge y \uparrow x \wedge y, \quad [8].\]
In every \((o)\)-continuous effect algebra if \( x_\alpha \overset{(o)}{\rightarrow} x \) and \( y_\alpha \overset{(o)}{\rightarrow} y \) then \( x_\alpha \lor y_\alpha \overset{(o)}{\rightarrow} x \lor y \) and \( x_\alpha \wedge y_\alpha \overset{(o)}{\rightarrow} x \wedge y \).

**Theorem 6.** In every complete \((o)\)-continuous effect algebra \( E \), for \( x_\alpha, x, y \in E \):

(i) \( x_\alpha \overset{\tau_o}{\rightarrow} x \implies x_\alpha \lor y \overset{\tau_o}{\rightarrow} x \lor y \)

(ii) \( x_\alpha \overset{\tau_o}{\rightarrow} x \implies x_\alpha \land y \overset{\tau_o}{\rightarrow} x \land y \)

(iii) \( x_\alpha \overset{\tau_o}{\rightarrow} x \implies x'_\alpha \overset{\tau_o}{\rightarrow} x' \)

**Proof.** (i)–(iii) follow from \((o)\)-continuity of \( E \) using Lemma 5, (iv). \( \square \)
Definition 7. A complete effect algebra $E$ is $(o)$-topological (order topological) if $(o)$-convergence of nets of elements coincides with $\tau_o$-convergence and $E$ is $(o)$-continuous.

An element $u$ of an effect algebra $E$ is called finite if there is a finite sequence $\{p_1, \ldots, p_n\}$ of not necessarily different atoms of $E$ such that $u = p_1 \oplus p_2 \oplus \cdots \oplus p_n$. If $E$ is complete then for every $x \in E$ we have $x = \bigvee \{u \in E \mid u \leq x, u \text{ is finite}\}$, by [25, Theorem 3.1]. If $E$ is complete atomic and $(o)$-continuous then the join of two finite elements is finite as well [21, Theorem 4.4].

Theorem 8. A complete atomic $(o)$-continuous effect algebra $E$ is $(o)$-topological iff $\tau_o$ on $E$ is Hausdorff.

Proof. (1) Assume that $\tau_o$ on $E$ is Hausdorff and for $x, x_a \in E$ let $x_a \xrightarrow{\tau_o} x$, $\alpha \in \mathcal{E}$. If $a \in E$ is an atom such that $a \leq x$ then by Theorem 6 we have $x_a \wedge a \xrightarrow{\tau_o} x \wedge a = a$. It follows that there is $\alpha_a \in \mathcal{E}$ such that $a \leq x_{\alpha_a}$ for all $\alpha \geq \alpha_a$, since otherwise there is a cofinal $\mathcal{E}' \subseteq \mathcal{E}$ such that $x_{\alpha_a} \wedge a \xrightarrow{\tau_o} 0$, $\alpha \in \mathcal{E}'$. By [16], we obtain that $x_a \ominus a \xrightarrow{\tau_o} x \ominus a$, $\alpha \geq \alpha_a$. By the same argument, for an atom $b \leq x \ominus a$ there is $\alpha_b \geq \alpha_a$ such that for all $\alpha \geq \alpha_b$ we have $b \leq x_{\alpha} \ominus a$ which gives $a \oplus b \leq x_{\alpha}$. By induction, for every finite element $u = a_1 \ominus a_2 \ominus \cdots \ominus a_n \leq x$, where $a_1, \ldots, a_n \in E$ are not necessary different atoms, there is $\alpha_u \in \mathcal{E}$ such that for all $\alpha \geq \alpha_u$ we have $u \leq x_{\alpha}$ and hence $u \leq \bigwedge_{\alpha \geq \alpha_u} x_{\alpha}$. We obtain that $x \leq \bigvee_{\beta \in \mathcal{E}} \bigwedge_{\alpha \geq \beta} x_{\alpha}$, because $x = \bigvee \{u \in E \mid u \leq x, u \text{ is finite}\}$. Further, $x_a \xrightarrow{\tau_o} x \implies x_a' \xrightarrow{\tau_o} x'$, which gives $x' \leq \bigvee_{\beta \in \mathcal{E}} \bigwedge_{\alpha \geq \beta} x_{\alpha}'$. By D’Morgan laws we obtain $\bigwedge_{\beta \in \mathcal{E}} \bigvee_{\alpha \geq \beta} x_{\alpha} \leq x$. We conclude that $x = \bigvee_{\beta \in \mathcal{E}} \bigwedge_{\alpha \geq \beta} x_{\alpha}$ which is equivalent to $x_a \xrightarrow{(o)} x$, because

$$x \uparrow \bigwedge_{\alpha \geq \beta} x_{\alpha} \leq x \leq \bigvee_{\alpha \geq \beta} x_{\alpha} \downarrow x.$$

(2) If $E$ is $(o)$-topological then $\tau_o$ on $E$ is Hausdorff, as the $(o)$-limit of an $(o)$-convergent net is unique. \qed

Theorem 9. Let $E$ be a complete atomic $(o)$-topological effect algebra. Then

(i) For every atom $a$ of $E$ the intervals $[a, 1]$ and $[0, a']$ are $\tau_o$-clopen sets.

(ii) For every two finite elements $u, v \in E$ the intervals $[u, 1]$, $[0, a']$ and $[u, v']$ are $\tau_o$-clopen sets.

(iii) Every $x \in E$ has a neighborhood base consisting of $\tau_o$-clopen sets $[u, v']$, $u, v$ finite.

Proof. (i) Evidently, $[a, 1]$ and $[0, a']$ are $\tau_o$-closed, since $\tau_o \subseteq \tau_o$. Let $x_a \xrightarrow{\tau_o} x$, for $\alpha \in \mathcal{E}$ and $x \in [0, 1]$. By Theorem 6, $x_a \wedge a \xrightarrow{\tau_o} x \wedge a = a$ and since $\tau_o$ is
Hausdorff, there is $\alpha_0 \in \mathcal{E}$ such that for all $\alpha \geq \alpha_0$ we have $x_\alpha \wedge a = a$, as otherwise $x_\alpha \wedge a \nrightarrow 0$. It follows that $x_\alpha \in [a, 1]$, for all $\alpha \geq \alpha_0$, which gives that $[a, 1]$ is open and hence also $[0, a']$ is open because $x \in [a, 1]$ if and only if $x' \in [0, a']$.

(ii) If $u = a_1 \oplus a_2 \oplus \cdots \oplus a_n$ where $a_k$ are atoms of $E$ and $x_\alpha \nrightarrow x \in [u, 1]$ then $a_1 \wedge x_\alpha \nrightarrow a_1 \wedge x = a_1$ and hence $a_1 \leq x_\alpha$ for all $\alpha \geq \alpha_1$. By [16, Theorem 3.3] for $\alpha \geq \alpha_1$ we have $x_\alpha \oplus a_1 \nrightarrow x \oplus a_1$. Since $a_2 \leq x \oplus a_1$, there is $\alpha_2 \geq \alpha_1$ such that for $\alpha \geq \alpha_2$ we have $a_2 \leq x_\alpha \oplus a_1$ which gives $a_1 \oplus a_2 \leq x_\alpha$. By induction there is $\alpha_n \geq \alpha_k$, $k = 1, 2, \ldots, n - 1$ such that for all $\alpha \geq \alpha_n$ we have $u \leq x_\alpha$. This proves that $[u, 1]$ is $\tau_\alpha$-clopen. It follows that $[0, u']$ is $\tau_\alpha$-clopen for every finite $u \in E$. Thus for all finite $u, v \in E$ we have $[u, v'] = [u, 1] \cap [0, v']$ is $\tau_\alpha$-clopen.

(iii) Let $x \in E$ be arbitrary and $x \in U(x) \in \tau_\alpha$. Put $P_x = \{u \in E \mid u \leq x, u \text{ is finite}\}$ and $Q_x = \{v \in E \mid v \leq x', v \text{ is finite}\}$. Then $x = \bigvee P_x$ and $x' = \bigvee Q_x$. For every finite set $F \subseteq P_x \cup Q_x$ we put $u_F = \bigvee (F \cap P_x)$ and $v_F = \bigvee (F \cap Q_x)$. Evidently $E = \{F \subseteq P_x \cup Q_x \mid F \text{ is finite}\}$ is directed by set inclusion and $u_F \uparrow x$, $v_F \uparrow x'$ which gives $v_F' \downarrow x$. Since $x \in U(x) \in \tau_\alpha$ there is $F_0 \in E$ such that $[u_{F_0}, v_{F_0}'] \in U(x)$ (see, e.g., Appendix B by H. Kirchheimová and Z. Riečanová, Proposition B.2.1 in [19]) and the interval $[u_{F_0}, v_{F_0}']$ is $\tau_\alpha$-clopen, as $u_{F_0}$ and $v_{F_0}$ are finite elements of $E$ ([21, Theorem 4.4]).

A direct product of a family $\{E_\kappa \mid \kappa \in H\}$, $H \neq \emptyset$, of effect algebras is the effect algebra $(\hat{E}; \oplus, 0, 1)$, where $\hat{E} = \prod \{E_\kappa \mid \kappa \in H\}$ is a Cartesian product and all operations $\oplus$, $0$, $1$ are defined componentwise. It follows that also the partial order and lattice operations (for lattice ordered $E_\kappa$, $\kappa \in H$) in $\hat{E}$ are defined componentwise.

Recall that effect algebras $(E; \oplus_E, 0_E, 1_E)$ and $(F; \oplus_F, 0_F, 1_F)$ are isomorphic if there exists a bijective map $\varphi : E \rightarrow F$ such that

(i) $\varphi(1_E) = 1_F$,

(ii) for all $a, b \in E$: $a \leq b'$ if and only if $\varphi(a) \leq \varphi(b')$,

in which case $\varphi(a \oplus_E b) = \varphi(a) \oplus_F \varphi(b)$.

We write $E \cong F$. Sometimes we identify $E$ with $\varphi(E)$.

In every complete effect algebra $E$ the center $C(E) = \{z \in E \mid z \wedge z' = 0 \}$ and $z \leftrightarrow x$ for all $x \in E$ is a complete Boolean algebra and $S(E) = \{z \in E \mid z \wedge z' = 0\}$ is a complete orthomodular lattice, and both are full sublattices of $E$, as we have shown in [11] and [25]. It follows that $C(E)$ and $S(E)$ are $\tau_0$-closed subsets of $E$, since they evidently contain all $(o)$-limits of their $(o)$-convergent nets. In view of Lemma 5, (iii), $C(E)$ and $S(E)$ are $(o)$-topological. It follows that $C(E)$ and $S(E)$ are atomic and $(o)$-continuous [4, Lemma 2.2]. We have shown [18, Lemma 4.3] that then $E$ can be decomposed into direct product of irreducible effect algebras.

Proposition 10. In every $(o)$-topological effect algebra $E$:...
(i) The center $C(E)$ is a complete atomic Boolean algebra.

(ii) The set of sharp elements $S(E)$ is a complete atomic and $(o)$-continuous orthomodular lattice.

(iii) $E \cong \prod \{[0, p] \mid p$ is an atom of $C(E)\}$, $[0, p]$ are irreducible effect algebras.

Note that the atomicity of $C(E)$ and $S(E)$ for a complete effect algebra $E$ does not imply that $E$ is atomic.

Example 11. Let $E = [0, 1] \subseteq R$ with defined $a \oplus b = a + b$ iff $a + b \leq 1$. $E$ is a complete MV-effect algebra, since every pair of elements of $E$ are comparable and hence compatible. Clearly $E$ has separated intervals which implies that $\tau_i$ on $E$ is Hausdorff and hence $\tau_o = \tau_i$ is a compact Hausdorff topology on $E$ (see [4] and [17]). Moreover, as $E$ is complete, for $x_\alpha \in E$, $\alpha \in E$ we have $x_\alpha (o) x$ iff $x = \bigvee_{\beta \in \mathcal{E}} \bigwedge_{\alpha \geq \beta} x_\alpha = \bigwedge_{\beta \in \mathcal{E}} \bigvee_{\alpha \geq \beta} x_\alpha$ iff $x \xrightarrow{\tau_o} x$, hence $(o)$-convergence is a topological convergence on $E$. Evidently $E$ is $(o)$-continuous. Note that $C(E) = S(E) = \{0, 1\}$.

3. Compact $(o)$-topological effect algebras

Definition 12. A complete effect algebra is compact $(o)$-topological if $E$ is $(o)$-topological and $\tau_o$ is compact.

The interval topology $\tau_i$ on a bounded lattice $L$ is the topology for which an open base is generated by complements of finite unions of closed intervals. It follows that $\tau_i \subseteq \tau_o$. If $\tau_i$ is Hausdorff then $\tau_i = \tau_o$ (see [5]). By [7], $\tau_i$ on $L$ is compact iff $L$ is a complete lattice. Thus on every complete effect algebra $E$ with Hausdorff interval topology the order topology $\tau_o$ is compact and Hausdorff. Nevertheless, such an effect algebra $E$ need not be $(o)$-topological since $E$ need not be $(o)$-continuous (see Example 20).

Definition 13. A bounded lattice $L$ has separated intervals, if given any two disjoint intervals $[a, b], [c, d] \subseteq L$, the lattice $L$ can be covered by a finite number of closed intervals each of which is disjoint with at least one of the intervals $[a, b]$ and $[c, d]$.

We have shown in [17, Lemma 2.2] that the interval topology on a complete lattice $L$ is Hausdorff iff $L$ has separated intervals.

Theorem 14. Every complete atomic $(o)$-continuous effect algebra with separated intervals is compact $(o)$-topological.

Proof. Since $\tau_i$ is Hausdorff, we have $\tau_o = \tau_i$ and hence $\tau_o$ is compact Hausdorff topology, as $E$ is complete. By Theorem 8, $E$ is $(o)$-topological. □

Lemma 15. Let $\{E_\kappa \mid \kappa \in H\}$ be a family of complete lattice effect algebras.
such that 

Theorem 9. As 

Because 

We have proved in [25, Theorem 3.1] that 

if for every orthogonal 

Let 

Theorem 18. 

Separated intervals and it is compact 

Corollary 17. 

In view of Lemma 15, 

Every complete atomic distributive effect algebra 

is 

is Hausdorff if 

(ii) This follows by (i) and [17, Lemma 2.2] 

Theorem 16. Every complete atomic distributive effect algebra 

has separated intervals and it is compact 

(o)-topological with 

\( \tau_o = \tau_i \). 

Proof. By [22, Theorem 3.1 and Corollary 3.2] we have 

\( E \cong \prod \{ [0, p_\kappa] \mid \kappa \in H \} \)

where 

\( \{ p_\kappa \mid \kappa \in H \} \) is the set of all atoms of \( C(E) \) for every \( \kappa \in H \) the interval \( [0, p_\kappa] \) is either a finite chain or a distributive diamond \( \{ 0_\kappa, a_\kappa, b_\kappa, p_\kappa \} \) in which 

\( p_\kappa = 2a_\kappa = 2b_\kappa \), hence all \( [0, p_\kappa] \) have separated intervals. Moreover, 

\( E \) is \( (o) \)-continuous. By Theorem 8, it follows that 

\( E \) is \( (o) \)-topological since, in view of Lemma 15, \( \tau_o = \tau_i \) is compact Hausdorff. 

Corollary 17. Every complete atomic MV-effect algebra \( (MV\text{-algebra}) E \) has separated intervals and it is compact 

(o)-topological with 

\( \tau_o = \tau_i \). 

Note that statements of Theorem 16 and Corollary 17 need not be true for non-atomic \( E \), as no non-atomic complete Boolean algebra is \( (o) \)-topological. On the other hand there are non-atomic MV-effect algebras which are compact 

(o)-topological with \( \tau_o = \tau_i \) (see Example 11). 

The set \( A_E \) of all atoms of an atomic effect algebra \( E \) is called almost orthogonal if for every \( p \in A_E \) the set \( \{ a \in A_E \mid a \nleq p \} \) is finite. 

Theorem 18. Let \( E \) be a compact \( (o) \)-topological atomic effect algebra and \( A_E \) is the set of all atoms of \( E \). Then: 

(i) \( A_E \) is almost orthogonal. 

(ii) For every \( p \in A_E \) there are finite elements \( u_1, u_2, \ldots, u_n \) of \( E \) such that 

\( E = (\bigcup_{k=1}^n [u_k, 1]) \cup [0, p'] \) and 

\( [0, p'] \cap (\bigcup_{k=1}^n [u_k, 1]) = \emptyset \). 

Proof. Set \( A_E = \{ p \in E \mid p \text{ is an atom of } E \} \). Let \( p \in A_E \) and \( x \in E \), \( x \neq 0 \). We have proved in [25, Theorem 3.1] that 

\( x = \bigvee \{ u \in E \mid u \leq x, u \text{ is finite} \} \). 

It follows that either \( x \leq p' \), or there is a finite element \( u \in E \) with \( u \leq x \) and 

\( u \nleq p' \). Let \( U = \{ u \in E \mid u \text{ is finite} \} \). Then 

\( E = (\bigcup_{u \in U, u \nleq p'} [u, 1]) \cup [0, p'] \). 

Because \( E \) is \( (o) \)-topological, the intervals \( [u, 1] \) and \( [0, p'] \) are \( \tau_o \)-clopen, by Theorem 9. As \( E \) is compact there is a finite set \( \{ u_1, u_2, \ldots, u_n \} \subseteq U \), 

\( u_k \nleq p' \) such that 

\( E = (\bigcup_{k=1}^n [u_k, 1]) \cup [0, p'] \). Evidently 

\( (\bigcup_{k=1}^n [u_k, 1]) \cap [0, p'] = \emptyset \).
as $u_k \not\leq p'$, for $k = 1, \ldots, n$. Moreover, if $a \in A_E$ and $a \not\leq p'$ then there is $k \in \{1, \ldots, n\}$ such that $a = u_k$. It follows that the set $\{a \in A_E \mid a \not\leq p'\}$ is finite. □

**Proposition 19.** There are nonatomic complete MV-effect algebras which are compact $(o)$-topological with $\tau_o = \tau_i$ (see Example 11).

The next example shows that a complete atomic effect algebra $E$ with compact Hausdorff order topology and $\tau_i = \tau_o$ need not be (compact) $(o)$-topological since $E$ need not be $(o)$-continuous.

**Example 20.** Let $E$ be a horizontal sum of MV-effect algebras $M_1$ and $M_2$, which means that we identify least and greatest elements of $M_1$ and $M_2$, respectively, and all pairs $a \in M_1 \setminus \{0, 1\}$ and $b \in M_2 \setminus \{0, 1\}$ are noncomparable. Further, let $M_1$ be an MV-effect algebra derived from a Boolean algebra with infinitely many atoms and $M_2 = \{0, a, 2a = 1\}$.

To show that $E$ is not $(o)$-continuous, we put $A = \{p \in M_1 \mid p$ is an atom of $M_1\}$ and $u_\alpha = \bigvee \alpha$, for all finite sets $\alpha \subseteq A$. Then $u_\alpha \uparrow 1$, but $u_\alpha \land a = 0$ while $1 \land a = a$. It follows that $E$ is not $(o)$-topological. Clearly $E$ has separated intervals which gives that $\tau_o = \tau_i$ is Hausdorff compact topology, as $E$ is complete.

Finally, the next example shows that an $(o)$-topological complete atomic modular effect algebra $E$ need not be compact $(o)$-topological.

**Example 21.** Let the effect algebra $E$ be a horizontal sum (0–1 pasting) of countably many distributive diamonds $E_\kappa = \{0_\kappa, a_\kappa, b_\kappa, 1_\kappa = 2a_\kappa = 2b_\kappa\} \kappa \in H$ and $H$ be infinite. Evidently, $\tau_o$ is discrete and hence it is not compact since $H$ is infinite.

**References**


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