A MULTIPLICITY RESULT FOR A GENERALIZED PENDULUM EQUATION

BORIS RUDOLF

Department of Mathematics, Faculty of Electrical engineering STU, 812 19 Bratislava, Slovakia

We consider the second order differential equation

\[ x'' + cx' = f(t, x, x') \]  

with the periodic boundary conditions

\[ x(a) = x(b), \quad x'(a) = x'(b) \]  

where \( f : I = [a, b] \times \mathbb{R}^2 \to \mathbb{R} \) is a continuous bounded function, \( c \neq 0 \).

Our goal is to prove two types of existence results for the problem (1), (2) using a method of lower and upper solutions similarly as it is done in [4], [5] for the equation \( x'' = f(t, x, x') \) with a bounded nonlinearity.

Rachůnková in [3] has proved the existence results with one sided linear growth conditions on \( f \) for the case when a lower solution is greater then an upper one. This result can be applied on (1), (2) when \( |c| < \frac{1}{b-a} \). Our result need no restriction on a constant \( c \).

Theorem 1 handle with the case when a lower solution is less then an upper one and is proved already in a more general situation when \( f - cx' \) satisfies a Nagumo–Bernstein condition [1], [4]. We give our proof only for completeness.

We apply the existence results given in Theorems 1 and 2 to prove a multiplicity result for a problem (1), (2) with a periodic nonlinearity.

Definition. The function \( \alpha(t) \) is called a lower solution for the problem (1), (2) if

\[ \alpha''(t) + c\alpha'(t) \geq f(t, \alpha(t), \alpha'(t)) \], \[ \alpha(a) = \alpha(b), \quad \alpha'(a) = \alpha'(b) \].

Similarly the function \( \beta(t) \) is called an upper solution for the problem (1), (2) if

\[ \beta''(t) + c\beta'(t) \leq f(t, \beta(t), \beta'(t)) \], \[ \beta(a) = \beta(b), \quad \beta'(a) = \beta'(b) \].

If the strict inequalities hold \( \alpha, \beta \) are called strict lower and upper solutions.
Lemma 1. Let $\alpha$, $\beta$ be a strict lower and upper solutions and $u(t)$ be a solution of the problem (1), (2).

Then $\alpha(t) \leq u(t)$ implies $\alpha(t) < u(t)$ and $\beta(t) \geq u(t)$ implies $\beta(t) > u(t)$.

Proof. Let $\beta(t) \geq u(t)$ and $0 = u(t_0) - \beta(t_0)$ at $t_0 \in (a,b)$. Then
\[
0 = u(t_0)^\prime - \beta(t_0)^\prime = u(t_0)^\prime - \beta(t_0)^\prime + cu(t_0)^\prime - c\beta(t_0)^\prime = f(t_0, u(t_0), u(t_0)) - \beta(t_0)^\prime - c\beta(t_0)^\prime \geq f(t_0, \beta(t_0), \beta(t_0)) - \beta(t_0)^\prime - c\beta(t_0)^\prime > 0,
\]
a contradiction.

Let $0 = u(a) - \alpha(a)$, $u(t) < \beta(t)$ for $t \in (a,b)$. Then $u'(a) = \beta'(a)$ and we obtain the same contradiction as above. \qed

Let $X = C^1(I)$, $\text{dom}L = \{x(t) \in C^2(I), \ x \text{ satisfies (2)}\}$, $Z = C(I)$. We denote
\[
L : \text{dom}L \subset X \rightarrow Z, \quad Lx = x'' + cx',
\]
\[
N : X \rightarrow Z, \quad Nx(t) = f(t, x(t), x'(t)).
\]
The problem (1), (2) is equivalent to the operator equation
\[
Lx = Nx,
\]
where the operator $N$ is $L$-compact [1].

We denote
\[
\Omega_{r,\rho} = \{x(t) \in C^1(I), \ ||x|| < r, \ ||x' + cx|| < \rho\}.
\]

Lemma 2. Let
(i) there is a constant $r > 0$ such that $f(t, r, 0) > 0$ and $f(t, -r, 0) < 0$,
(ii) $|f(t, x, y)| \leq M$,

Then there is $\rho_0 > 0$ such that the topological degree
\[
D(L, N, \Omega_{r,\rho}) = 1 \pmod{2}
\]
for each $\rho > \rho_0$ i.e. there is a solution $x(t)$ of (1), (2) such that $|x(t)| < r$, $|x'(t) + cx(t)| < \rho$.

Proof.
We consider the homotopy
\[
Lx = \tilde{N}(x, \lambda)
\]
defined by the parametric system of equations
\[
x'' + cx' = \lambda f(t, x, y) + (1 - \lambda)x, \quad (6)
x(a) = x(b), \quad x'(a) = x'(b). \quad (2)
\]
Now $-r, r$ are a strict lower and upper solutions of the problem (6).

As $|\lambda f(t, x, y) + (1 - \lambda)x| \leq M + r$, then for each solution of (6) such that $|x(t)| \leq r$ there is $|x'(t) + cx(t)| \leq \frac{b - a}{2} (M + r) = \rho_0$.

The above estimation and Lemma 1 imply that no solution of (6), (2) lies on the boundary of $\partial \Omega_{r,\rho}$, $\rho \geq \rho_0$.

By the generalized Borsuk theorem [2]
\[
D(L, \tilde{N}(, 1), \Omega_{r,\rho}) = D(L, \tilde{N}(, 0), \Omega_{r,\rho}) = 1 \pmod{2}
\]
and Lemma 2 is proved. \qed
We show that \( \alpha \) upper solution of (7), (2). Lemma 1 implies \( u \).

Proof. i.e. there is a solution \( x \).

Theorem 1. Let

(i) \( \alpha \) be a strict lower and upper solutions of the problem (1), (2).

(ii) \( |t| \leq M \), for each \( t \in I \) \( \alpha \leq \beta \), \( y \in R \).

Then there is a constant \( \rho_0 \) such that for each \( \Omega_1 = \{ x(t) \in C^1(I), \alpha(t) < x(t) < \beta(t), \| x' + cx \| < \rho \} \), \( \rho > \rho_0 \) there is

\[
D(L, N, \Omega_1) = 1 \quad (\text{mod } 2)
\]

i.e. there is a solution \( x(t) \in \Omega \) of (1), (2).

Proof.

Let \( r = \max \{|\alpha|, |\beta|\} \).

We define a perturbation

\[
f^*(t, x, y) = \begin{cases} 
  f(t, \beta(t), y) + M(r - \beta(t)) + M & x > r + 1, \\
  f(t, \beta(t), y) + M(x - \beta(t)) & \beta(t) < x \leq r + 1, \\
  f(t, x, y) & \alpha(t) \leq x \beta(t), \\
  f(t, \alpha(t), y) - M(\alpha(t) - x) & -r - 1 \leq x < \alpha(t), \\
  f(t, \alpha(t), y) - M(\alpha(t) + r) & x < -r - 1.
\end{cases}
\]

Then \( |f^*| \leq 2M \) and the assumptions of Lemma 2 are satisfied for \( \Omega_{r+1, \rho}, \rho > \rho_0 \) where \( \rho_0 = \frac{b - a}{2}(2M + r + 1) \).

Suppose \( u(t) \in \Omega_{r+1, \rho} \) is a solution of the problem

\[
x'' + cx' = f^*(t, x, x'), \quad (2),
\]

\[
x(a) = x(b) \quad x'(a) = x'(b).
\]

We show that \( \alpha \leq u \leq \beta \).

Let \( v(t) = u(t) - \beta(t) \) attains its maximum \( v_{\text{max}} > 0 \). Then \( \beta(t) + v_{\text{max}} \) is a strict upper solution of (7), (2). Lemma 1 implies \( u(t) < \beta(t) + v_{\text{max}} \) a contradiction.

That means \( u(t) \) is a solution of (1), (2).

Then

\[
D(L, N^*, \Omega_{r+1, \rho}) = D(L, N^*, \Omega_1) = D(L, N, \Omega_1) = 1 \quad (\text{mod } 2). \quad \square
\]

Now we assume that a lower and upper solutions are in a more general position.

Theorem 2. Let

(i) \( |f(t, x, y)| < M \),

(ii) \( \alpha, \beta, \alpha(t) \leq \beta(t) \), be a strict lower and upper solutions for the problem (1), (2).

Then there are constants \( r, \rho_0 > 0 \) such that

\[
D(L, N, \Omega_2) = 1 \quad (\text{mod } 2)
\]

where \( \Omega_2 = \{ x(t) \in C^1(I), \exists t_\alpha, t_\beta \in I, \beta(t_\beta) < x(t_\beta), x(t_\alpha) < \alpha(t_\alpha), \| x \| < r, \| x' + cx \| < \rho \} \), \( \rho > \rho_0 \).
there is a solution $x(t) \in \Omega_2$ of the problem (1), (2).

**Proof.** Let $r = \max (|\alpha|, |\beta|) + \frac{(b-a)}{c}M$.

We define a perturbation $f^*$ by

$$f^*(t, x, y) = \begin{cases} 
  f(t, x, y) + M & x > r + 1, \\
  f(t, x, y) + M(x - r) & r < x \leq r + 1, \\
  f(t, x, y) - M & x < -r - 1.
\end{cases}$$

Clearly $r + 1, -r - 1$ are a strict lower and upper solutions of the problem

$$x'' + cx' = f^*(t, x, x'), \quad (8)$$

$$x(a) = x(b) \quad x'(a) = x'(b). \quad (2)$$

As $|f^*| < 2M$ then for each solution of (8) the boundary conditions (2) imply $|x'(t) + cx(t)| \leq (b-a)M$. Then $\max |x(t)| \leq \frac{(b-a)M}{c}$. Set $\rho_0 = \frac{(b-a)}{2}(2M + r + 1)$.

Then for $\rho > \rho_0$

$$D(L, N^*, \Omega_{r+1, \rho}) = 1 \quad (\text{mod 2})$$

Let now

$$\Omega_l = \{x(t) \in \Omega_{r+1, \rho}, \quad -r - 1 < x < \beta\},$$

$$\Omega_u = \{x(t) \in \Omega_{r+1, \rho}, \quad \alpha < x < r + 1\}.$$}

Then

$$D(L, N^*, \Omega_l) = D(L, N^*, \Omega_u) = 1 \quad (\text{mod 2})$$

Set $\Omega_m = \Omega_{r+1, \rho} \setminus (\Omega_l \cup \Omega_u)$.

As $-r - 1, \alpha, r + 1, \beta$ are strict lower and upper solutions, Lemma 1 implies there is no solution $u \in \partial \Omega_m$.

The addition property of the degree means

$$D(L, N^*, \Omega_m) = 1 \quad (\text{mod 2})$$

and finally the excision property implies

$$D(L, N^*, \Omega_m) = D(L, N^*, \Omega_2) = D(L, N, \Omega_2) = 1 \quad (\text{mod 2}). \quad \square$$
We apply the previous results to a periodic boundary value problem for a generalized oscillator

\[ x'' + cx' = f(t, x, x') \]  
\[ x(a) = x(b) \quad x'(a) = x'(b), \]

assuming that the function \( f \) is \( 2\pi \) periodic in variable \( x \).

Assume that there are \( \alpha(t), \beta(t) \) a strict lower and upper solutions of (1), (2). The periodicity of \( f \) implies that \( \alpha(t) + 2k\pi, \beta(t) + 2k\pi \) are again a strict lower and upper solutions of (1), (2) for each \( k \in \mathbb{Z} \).

Then there is a \( k \in \mathbb{Z} \) such that \( \alpha(t) + 2k\pi < \beta(t) \) and \( \alpha(t) + (2k + 1)\pi \not\equiv \beta(t) \).

Then Theorem 1 and Theorem 2 imply there are two different families \( x_1(t) + 2k\pi, x_2(t) + 2k\pi \) of solutions of the problem (1), (2).

References